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Renormalization group and logarithmic corrections to scaling relations in conformal sector of 4D gravity

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Abstract

We study the effective theory of the conformal factor near its infrared stable fixed point. The renormalization group equations for the effective coupling constants are found and their solutions near the critical point are obtained, providing the logarithmic corrections to the scaling relations. Some cosmological applications of the running of coupling constants are briefly discussed.

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It is quite an old idea to work with some effective theory for quantum gravity (QG) in the absence of a consistent theory. The traditional candidate for such an effective theory is usually Einstein gravity which is known to be non-renormalizable [1]. Despite this fact, already in this theory the loop corrections can be calculated and may become quite relevant for a variety of phenomena [2, 3] (for a general introduction to perturbative QG, see for example [4]). Recently, an interesting effective model aiming to describe QG in the far infrared, has been introduced [5]. This model is based on the effective theory for the conformal factor, and some of its properties have been further studied in refs.[6]-[9]. It was found that it possesses a non-trivial infrared (IR) stable fixed point which could become physically relevant at cosmological distance scales. Investigation of this critical point led to the derivation of exact scaling relations and, in particular, of the anomalous dimension of the conformal factor in analogy with 2D non-critical string theory. Furthermore, this model provides a dynamical framework for a solution of the cosmological constant problem [5].

The purpose of this letter is to study the effective theory of the conformal factor near its fixed point. We show that making the one-loop renormalization in a way which preserves global conformal invariance of counterterms, one can rigorously construct renormalization group (RG) equations for coupling constants. The solution of these equations near the critical point gives the logarithmic corrections to scaling relations.

Consider the flat space theory described by the Lagrangian

$$\mathcal{L} = b_1 \left(\Box\sigma\right)^2 + b_2 \left(\partial_\mu\sigma\right)^2 \Box\sigma + b_3 \left[\left(\partial_\mu\sigma\right)^2\right]^2 + b_4(\sigma)\left(\partial_\mu\sigma\right)^2 + b_5(\sigma) \tag{1}$$

where σ has classically canonical dimension zero, b_1, b_2, b_3 are dimensionless coupling constants and b_4, b_5 are for the moment arbitrary dimensionful field-dependent generalized couplings. The one-loop divergences of this theory were computed in ref.[9] and for some choices of b_4, b_5 it is multiplicatively renormalizable in the usual sense. These choices correspond to the effective theory of the conformal factor [5] which we review below.

We start for simplicity with N conformally invariant scalars in curved spacetime. The conformal anomaly in this case is [10]

$$T^\mu_\mu = b\left(F + \frac{2}{3}\Box R\right) + b'G + b''\Box R, \tag{2}$$

where

$$b = \frac{N}{120(4\pi^2)}, \quad b' = -\frac{N}{360(4\pi^2)},$$

and b'' may be changed by the variation of a local R^2 counterterm in the gravitational part of the action. In eq.(2) R is the scalar curvature, F is the square of Weyl tensor and G is the Gauss-Bonnet combination. Choosing the conformal parametrization $g_{\mu\nu}(x) = e^{2\sigma}\bar{g}_{\mu\nu}$ and integrating over σ, we get an anomaly induced action [11, 5]. Adding to this action the classical Einstein action in conformal parametrization we get the effective theory for the conformal factor. In notations of ref.[5] and in flat background metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the resulting Lagrangian is

$$\mathcal{L} = -\frac{Q^2}{(4\pi)^2}(\Box\sigma)^2 - \zeta\left[2\alpha(\partial_\mu\sigma)^2\Box\sigma + \alpha^2(\partial_\mu\sigma)^4\right] + \gamma e^{2\alpha\sigma}(\partial_\mu\sigma)^2 - \frac{\lambda}{\alpha^2}e^{4\alpha\sigma}, \qquad (3)$$

where $\zeta = b + 2b' + 3b''$, $\frac{Q^2}{(4\pi)^2} = \zeta - 2b'$, $\gamma = \frac{3}{\kappa}$ and $\lambda = \frac{\Lambda}{\kappa}$ with $\kappa = 8\pi G$. ζ is the coupling of the local R^2 term, γ and λ are the Newtonian and the cosmological couplings, respectively, while Q^2 at $\zeta = 0$ was interpreted as a four-dimensional central charge for which a possible four-dimensional C-theorem could be applied [6, 12]. In eq.(3) we also allowed for a non-trivial anomalous scaling dimension α for the conformal factor $e^{\alpha\sigma}$; its classical value is $\alpha = 1$.

As it can be easily seen, the theory (3) belongs to the type (1) and it is multiplicatively renormalizable by simple power counting arguments. It has been studied in the IR stable fixed point $\zeta = 0$ where it was argued that it describes an infrared phase of QG [5]. Here, we will discuss the behavior of the effective couplings away from the fixed point in a spirit different from the discussion of ref.[9], using a renormalization procedure based on physical requirements. In fact, the Lagrangian (3) is the most general one in flat space, containing up to 4 space-time derivatives and being invariant under global conformal transformations, with the conformal factor $e^{\alpha\sigma}$ having canonical dimension 1 [5]. Global conformal symmetry is actually the remnant of full general covariance of the effective theory, once the background metric is fixed to the flat metric.

An inspection of the structure of divergences based on power counting arguments shows that the four-derivative sector gets renormalized by diagrams involving only ζ-vertices. Moreover, there are in general only two independent renormalizations which can be chosen to be those of ζ and α. There is no independent wave function renormalization because the coefficient b' multiplies a non-local term in the action when expressed in terms of the full metric. On the other hand, the two-derivative coupling γ and the cosmological coupling λ are renormalized by diagrams involving in addition to ζ-vertices, exactly one γ-vertex, and two γ-vertices or one λ-vertex, respectively. A simple counting of combinatoric factors of higher-point functions then shows that all such divergences are of the form $e^{2\alpha\sigma}(\partial_\mu\sigma)^2$

and $e^{4\alpha\sigma}$. It follows that theory (3) is renormalizable by redefining the couplings ζ, γ, λ and α.

The one-loop counterterms for the theory (1) have been calculated in ref.[9]. Specifying the results to the model (3), we get the following expression for the corresponding one-loop divergences (in dimensional regularization):

$$\Gamma_{\text{div}} = -\frac{2}{\varepsilon}\int d^4x\sqrt{-g}\ \ \frac{(4\pi)^4}{Q^4}\left\{5\alpha^2\zeta^2\left[\Box\sigma+\alpha(\partial_\mu\sigma)^2\right]^2 \right. \\ \left. -\gamma\alpha^2\left(3\zeta+\frac{2Q^2}{(4\pi)^2}\right)(\partial_\mu\sigma)^2e^{2\alpha\sigma}+\left(\frac{8\lambda Q^2}{(4\pi)^2}-\frac{\gamma^2}{2}\right)e^{4\alpha\sigma}\right\} \tag{4}$$

It follows from expressions (4) and (3), that one-loop renormalization indeed preserves global conformal invariance in the structure of counterterms. Furthermore, in the four-derivative sector, only the coupling constant ζ should be renormalized at the one-loop level, while α remains a free parameter. The corresponding β-functions are:

$$\begin{aligned}\beta_\zeta &= \frac{80\pi^2\alpha^2\zeta^2}{Q^4}\\ \beta_\gamma &= \left[\frac{2\alpha^2}{Q^2}+3\zeta\alpha^2\frac{(4\pi)^2}{Q^4}\right]\gamma\\ \beta_\lambda &= \frac{8\lambda\alpha^2}{Q^2}-\frac{8\pi^2\alpha^2\gamma^2}{Q^4}\ .\end{aligned} \tag{5}$$

Now let us write the RG equations for the effective action. Their general form is:

$$\left\{\frac{\partial}{\partial t}-(\beta_p+pd_p)\frac{\partial}{\partial p}-(\gamma_\sigma+d_\sigma)\int d^nx\sigma(x)\frac{\delta}{\delta\sigma(x)}\right\}\Gamma[e^tk,\sigma,p,\mu]=0, \tag{6}$$

where Γ is the renormalized effective action, p stands for all the renormalized couplings $\{\zeta,\gamma,\lambda\}$, d_p and d_σ are the classical scaling dimensions for p and σ, and γ_σ is the anomalous dimension of σ. As we have seen above, $d_\sigma=\gamma_\sigma=0$. Finally k denotes the external momenta, and $t\to\infty$ $(-\infty)$ corresponds to the ultraviolet (infrared) limit.

The RG equations for the effective couplings (which determine the asymptotic behavior of the effective action Γ) are:

$$\frac{dp(t)}{dt}=\beta_p(t)+d_pp(t) \tag{7}$$

Using eq.(5) and the classical scaling dimensions of the various couplings p [5]: $d_\zeta=0$, $d_\gamma=2-2\alpha$, $d_\lambda=4-4\alpha$, the one-loop RG equations for the effective couplings become:

$$\frac{d\zeta(t)}{dt} = \frac{80\pi^2\alpha^2\zeta^2(t)}{Q^4(t)}, \tag{8}$$

$$\frac{d\gamma(t)}{dt} = \left[\frac{2\alpha^2}{Q^2(t)}+3\zeta(t)\alpha^2\frac{(4\pi)^2}{Q^4(t)}+2-2\alpha\right]\gamma(t), \tag{9}$$

$$\frac{d\lambda(t)}{dt} = \left[\frac{8\lambda(t)\alpha^2}{Q^2(t)}-\frac{8\pi^2\alpha^2\gamma^2(t)}{Q^4(t)}+(4-4\alpha)\lambda(t)\right]. \tag{10}$$

We remind that $Q^2(t)$ is not an independent function: $\frac{Q^2(t)}{(4\pi)^2} = \zeta(t) - 2b'$. It is also useful to define the dimensionless ratio $h \equiv \lambda/\gamma^2$ which satisfies the equation:

$$\frac{dh(t)}{dt} = \frac{4\alpha^2}{Q^2(t)} h(t) - \frac{8\pi^2\alpha^2}{Q^4(t)} - 6\frac{\alpha^2(4\pi)^2}{Q^4(t)}\zeta(t)h(t) \ . \tag{11}$$

The system of eqs.(8)-(11) was studied in the IR stable fixed point $\zeta = 0$ in ref.[5]. The anomalous scaling dimension of the conformal factor is determined by the vanishing of (9), while the fixed point of (11) determines the relation between the cosmological and Newtonian couplings. At one-loop level, the IR fixed point solutions are:

$$\zeta = 0, \quad \alpha = \alpha_0 \equiv \frac{1 - \left(1 - \frac{4}{Q_0^2}\right)^{1/2}}{2/Q_0^2}, \quad h = \frac{2\pi^2}{Q_0^2}, \quad \gamma = \text{arbitrary}, \tag{12}$$

where $Q_0^2 \equiv Q^2(\zeta = 0)$.

Now, the solution of (8) can be found only in non-explicit form:

$$\zeta^2 - 4b'\zeta \log\zeta - 4b'^2 = \frac{5\alpha^2\zeta}{(4\pi)^2} t + c\zeta \tag{13}$$

where c is an integration constant. The form of (13) is not very useful for obtaining analytic expressions for the solutions of RG equations. However, here we are interested in the asymptotic behavior of the solutions near the infrared fixed point (12). In this limit, eq.(13) gives:

$$\begin{aligned} \zeta(t) &\simeq -\frac{4b'^2(4\pi)^2}{5\alpha^2 t} \quad \to 0 \\ t &\to -\infty \end{aligned} \tag{14}$$

This is an asymptotically free solution in the infrared; its presence was also discussed in ref.[9], using a different form of renormalization of coupling constants. Using eq.(14), the solution of (9) and (11) in the infrared limit ($t \to -\infty$) is:

$$\begin{aligned} \gamma(t) &\simeq (-t)^{-1/5} e^{t\left(2 - 2\alpha + \frac{2\alpha^2}{Q_0^2}\right)}, \\ h(t) &\simeq \frac{2\pi^2}{Q_0^2} + e^{\frac{4\alpha^2}{Q_0^2}t} \ . \end{aligned} \tag{15}$$

Equations (14) and (15) give the leading logarithmic corrections to coupling constants in the infrared limit. Using the fixed point value $\alpha = \alpha_0$ from eq.(12), one finds that the Newtonian coupling γ decreases slowly at large distances as:

$$\gamma(t) \simeq (-t)^{-1/5} \ . \tag{16}$$

This result is specific to one-loop, since α remains unrenormalized at this level. Inclusion of higher order radiative corrections is expected in general to destabilize this behavior. In fact, for α away but close to its fixed point value (12), one gets an exponential growth or decay of γ, depending on whether α approaches α_0 from above or from below, respectively. Therefore, a two-loop computation is needed to clarify this point. On the other hand, the cosmological coupling behaves in all cases as the square of the Newtonian coupling. In fact, the ratio $h(t)$ approaches its asymptotic value exponentially fast.

One may consider a different physical interpretation of the above results, if one chooses the Planck scale as a unit of mass at all energies. In this context, the Newtonian coupling should be normalized to one, implying the vanishing of its derivative (9). In this way, we obtain an effective anomalous scaling dimension for the conformal factor:

$$\alpha(t) \simeq \alpha_0 + 4\pi^2\zeta(t)\left(-1 + (1 - 4/Q_0^2)^{1/2} + 2(1 - 4/Q_0^2)^{-1/2}\right). \tag{17}$$

The infrared running of the various couplings and in particular of Newton's constant may have interesting physical applications. Consider for instance the Newtonian potential which has the form:

$$V(r) = -\frac{Gm_1m_2}{r}, \tag{18}$$

where $G = 3/8\pi\gamma$. The Wilsonian gravitational potential, obtained from (18) by replacing G with the running Newton coupling, may have important consequences in cosmology, in particular to the dark matter problem [13]. Naturally, one has to use the RG parameter, $t = \frac{1}{2}\log\frac{r_0^2}{r^2}$, in analogy with the case of electrostatic potential in quantum electrodynamics $\left(\frac{\mu_0^2}{\mu^2} \sim \frac{r^2}{r_0^2}\right)$. As a result, we get

$$V(r) \simeq -\frac{G_0m_1m_2}{r}\left(1 - \frac{\alpha^2}{10Q_0^2}\log\frac{r_0^2}{r^2}\right), \tag{19}$$

where G_0 is the (initial) value of the Newton's constant at the distance r_0. Eq.(19) describes the leading-log corrections to the classical gravitational potential in the context of the effective theory of QG under investigation. Note that in Einstein action, the leading-log corrections are absent due to the trivial fact that the only coupling in this case is the dimensionful Newton's constant and loop corrections are fall off with powers of r^2.

It was suggested in ref.[13] that a possible logarithmic RG evolution of Newton's constant could lead to a scale dependence of density fluctuations with interesting effects to the dark matter problem. For this purpose, one needs the Newtonian coupling to be a

slowly growing function of distance, so that the density parameter Γ_0,

$$\Gamma_0 = \frac{8\pi}{3} \frac{G\rho_m}{H_0^2}, \tag{20}$$

has a similar behavior. In eq.(20) H_0 is the current value of the Hubble parameter, and ρ_m is the baryon matter density at the present epoch. This analysis was performed in the context of a higher-derivative R^2-gravity theory. In our case, eq.(16) implies that $G \sim G_0(-t)^{1/5}$ which suggests that the effective theory of conformal factor might be promising for such a direction.

Finally, let us present briefly a few remarks on the critical behavior of this system at non-zero temperature. Using the well-known analogy between finite-temperature and infrared properties of the corresponding three-dimensional theory, one may apply the technique of ϵ-expansion. For this purpose, it is enough to consider only the four-derivative sector of (3) (associated to dimensionless couplings in four dimensions), since all massive terms are irrelevant for this discussion. In $D = (4 - \epsilon)$ dimensions, one has to replace the dimensionless parameters Q^2 and ζ by $Q^2\mu^{-\epsilon}$ and $\zeta\mu^{-\epsilon}$ (the dimension of σ is not changing). The RG equation (8) for ζ is then modified by the addition of a new term $\epsilon\zeta(t)$ in the right hand side (at the end we put as usually $\epsilon = 1$). The fixed point solutions are defined by the equality of the new β-function for ζ to zero. As a result, we obtain three fixed points: $\zeta_1 = 0$, and $\zeta_{2,3} = 2b' - 5\alpha^2(2\epsilon(4\pi)^2)^{-1} \pm [(b' - 5\alpha^2(4\epsilon(4\pi)^2)^{-1})^2 - 4b'^2]^{1/2}$. Stability is imposed by the condition that the derivative of ζ β-function should be positive. It turns out that the fixed point $\zeta_1 = 0$ is IR stable, which is an indication that at this point the system undergoes a second order phase transition.

In summary, we discussed the effective theory of the conformal factor and found the solutions of RG equations near the critical point. As a result, we obtained the logarithmic corrections to scaling relations in the asymptotically free IR regime. We also discussed possible interesting cosmological applications of the running of Newtonian and cosmological couplings.

A weak point of theory (3) is the approximation of neglecting the spin-2 graviton modes [5]. It was argued [6] that their contribution might only lead to finite renormalization of the conformal anomaly coefficients at the fixed point, and it could be taken into account afterwards. In fact, the results of our RG consideration indicate that the effective theory of the conformal factor could be the approximate infrared limit of a more complete system of RG equations in the context of some consistent theory of QG where of course all degrees of freedom are taken into account. It would be interesting to develop such a point of view

further.

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